

Nonlocal phase-field systems with general potentials

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Abstract

We consider a phase-field model where the internal energy depends on the order parameter χ in a nonlocal way. Therefore, the resulting system consists of the energy balance equation coupled with a nonlinear and nonlocal ODE for χ . Such system has been analyzed by several authors, in particular when the configuration potential is a smooth double-well function. More recently, in the case of a potential defined on $(-1, 1)$ and singular at the endpoints, the existence of a finite-dimensional global attractor has been proven. Here we examine both the case of smooth potentials as well as the case of physically realistic (e.g., logarithmic) singular potentials. We prove well-posedness results and the eventual global boundedness of solutions uniformly with respect to the initial data. In addition, we show that the separation property holds in the case of singular potentials. Thanks to these results, we are able to demonstrate the existence of a finite-dimensional attractors in the present cases as well.

Key words: phase-field models, smooth and singular potentials, nonlocal operators, well-posedness, uniform regularization properties, finite dimensional global attractors.

AMS (MOS) subject classification: 35B41, 35Q79, 37L30, 80A22.

1 Introduction

A well-known approach to study two-phase Stefan-like problems in more than one spatial dimension is the so-called phase-field (or diffuse interface) method. Roughly speaking, it consists in introducing an order parameter χ whose zero level set substitutes for the sharp interface, while $\chi = \pm 1$ in the higher/lower energy phases. The classical problem is thus replaced by an order parameter dynamics, originated from the study of critical phenomena, coupled with the energy balance equation governing the temperature field. An important issue is to recover the original interface conditions and this is usually done by (formal) asymptotic expansions. Moreover, diffuse interface models are quite effective from the numerical viewpoint since there is no need for interface tracking. A significant and basic example of phase-field system is due to G. Caginalp (see [9], cf. also [8]), namely,

$$\vartheta_t + \frac{\ell}{2}\chi_t - k\Delta\vartheta = 0, \quad (1.1)$$

$$\tau\chi_t - \xi\Delta\chi + W'(\chi) = 2\vartheta, \quad (1.2)$$

on a given bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, for some time interval $(0, T)$. Here ϑ is a rescaled temperature so that $\vartheta = 0$ is the equilibrium melting temperature, while ℓ , k , τ and ξ are given positive

constants which represent the latent heat of fusion, the diffusivity, a relaxation time and a correlation length, respectively. The function W is (the density of) potential energy associated with the phase configuration. Such potential is usually a smooth double well function (typically $W(r) = \frac{1}{8}(r^2 - 1)^2$). However, this is just a convenient approximation of the physically more relevant *logarithmic potential* generally taking the form

$$W(r) = (1+r)\log(1+r) + (1-r)\log(1-r) - \gamma r^2, \quad \gamma \geq 0. \quad (1.3)$$

The mathematical literature on (1.1)-(1.2) is rather vast and we confine ourselves to quote the pioneering paper [14] and the more recent ones [17, 18] (see also references therein).

In order to analyze the microscopic influences of anisotropy on the interface, in [10] a phase-field system has been derived from microscopic considerations based on Statistical Mechanics. This system is similar to (1.1)-(1.2) but for the diffusion term $\xi \Delta \chi$ which is replaced by $A : D^2 \chi$, where $A \in \mathbb{R}^{d \times d}$ is positive definite and $D^2 \chi$ is the Hessian of χ . However, this derivation is performed by truncating the expansion of the interaction function (see [10, Prop. 2.4], cf. also [11] for higher-order approximations). Then the author, by using formal asymptotics, deduces a modified Gibbs-Thompson relation in 2D. More recently, by using the same procedure, a related phase-field model has been obtained without approximating the interaction function (see [12, 13]). Actually, working in a bounded domain and choosing $\lambda = 0$ in [13, Sec. 2], the system obtained there takes the following form:

$$\vartheta_t + \frac{\ell}{2} \chi_t - k \Delta \vartheta = 0, \quad (1.4)$$

$$\varsigma^2 \tau \chi_t - K_\varsigma * \chi + \kappa(x) \chi + W'(\chi) = 2\varsigma \vartheta. \quad (1.5)$$

Here $K_\varsigma(x) = \varsigma^{-d} K(\varsigma^{-1}x)$ where $\varsigma > 0$ is an atomic length scale and $K : \mathbb{R}^d \rightarrow \mathbb{R}$ is a sufficiently smooth interaction kernel satisfying $K(x) = K(-x)$ and such that $\kappa(x) := \int_\Omega K(x-y) dy$ is bounded and nonnegative.

The asymptotic limit $\varsigma \searrow 0$ has been analyzed in [13] and a new anisotropic interface condition has been obtained. On the other hand, this class of systems was already considered in some previous papers (cf., e.g., [2, 4, 5, 6, 15] and their references). In particular, rigorous mathematical results were proven for smooth potentials. Well-posedness for $\Omega = \mathbb{R}$ and $d = 1$ was established in [5] through semigroup theory. These results were then extended to bounded d -dimensional domains with either homogeneous Neumann or Dirichlet boundary conditions for ϑ (see [6, 15]). Regarding the longtime behavior, the convergence of a solution to a single stationary state was shown in [15] by means of a suitable nonsmooth version of the Łojasiewicz-Simon inequality. Existence of an absorbing set was proven in [6] as well as an analysis of the ω -limit sets. More recently, the results of [15] have been extended to a class of singular unbounded potentials which does not include the logarithmic ones (cf. [19]). Actually, in [19] equation (1.5) was modified by adding an inertial term of the form $\alpha \chi_{tt}$, $\alpha > 0$. However, the results proved there also hold for $\alpha = 0$. Then, by exploiting [19], the existence of a finite-dimensional global attractor has been established in [16]. Here we want to generalize such results to both smooth potentials and more general singular potentials (e.g., of the logarithmic type (1.3)). This goal is connected with the property of the solutions of getting bounded in finite time uniformly with respect to sufficiently general initial data. In addition, in the case of singular potentials, a (uniform) separation property is also needed. Here we prove all these properties for weak solutions originating from initial data in the energy space. In particular, in the case of smooth potentials, our results generalize the corresponding ones in [15]. Moreover, for singular potentials, the separation property holds instantaneously, namely for $t > 0$, even though the initial datum is a pure state (see Remark 4.2 below). As a consequence, we also demonstrate the existence of a finite dimensional global attractor using the approach devised in [16]. This approach exploits the only source of compactness for χ , namely $K_\varsigma * \chi$. Note that we cannot expect smoothing effects on χ .

For the sake of simplicity, we choose the constants in such a way that system (1.4)-(1.5) can be rewritten in the form

$$\vartheta_t + \chi_t - \Delta \vartheta = 0, \quad (1.6)$$

$$\chi_t + J[\chi] + \kappa(x) \chi + W'(\chi) = \vartheta, \quad (1.7)$$

in $\Omega \times (0, T)$. Here J is a linear operator which is a suitable generalization of the nonlocal convolution operator introduced above (see Section 2 below). Following [15], we endow the system with the following boundary and initial conditions

$$\vartheta = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (1.8)$$

$$\vartheta|_{t=0} = \vartheta_0, \quad \chi|_{t=0} = \chi_0 \quad \text{in } \Omega. \quad (1.9)$$

It is worth mentioning that there are also (mainly) existence and uniqueness results for more refined nonlocal phase-field systems, formulated with respect to the absolute temperature, which are thermodynamically consistent also far from the equilibrium temperature (cf. [22, 23, 24, 25, 30]). It would be interesting and challenging to extend some of the present results to such systems.

The plan of this paper goes as follows. The main results about well-posedness and regularization properties of the solutions are stated in Section 2. Then, the corresponding theorems for smooth potentials and singular potentials are proven in Section 3 and Section 4, respectively. The final Section 5 is devoted to the existence of the global attractor.

2 Well posedness and regularization results

In the sequel we will assume that $|\Omega| = 1$, for simplicity. We set $V := H_0^1(\Omega)$, $H := L^2(\Omega)$ and note by $\|\cdot\|$ the norm in H , by (\cdot, \cdot) the scalar product of H , and by $\|\cdot\|_p$ the norm in $L^p(\Omega)$ for $p \in [1, \infty]$. We also let A be the Laplace operator with homogeneous Dirichlet b.c., seen either as an unbounded linear operator on H with domain $V \cap H^2(\Omega)$ or as a bounded linear operator from V to its topological dual V' . Then (1.6)-(1.8) can be rewritten as follows:

$$\vartheta_t + \chi_t + A\vartheta = 0, \quad \text{in } V', \quad (2.1)$$

$$\chi_t + J[\chi] + f(\cdot, \chi) = \vartheta, \quad \text{in } H, \quad (2.2)$$

with

$$f(x, r) = f_0(r) - \lambda(x)r, \quad (2.3)$$

where $\lambda \in L^\infty(\Omega)$ is a given function, and

$$f_0 \in C^1(I; \mathbb{R}), \quad f'_0(r) \geq 0 \quad \forall r \in I, \quad f_0(0) = 0, \quad (2.4)$$

where I , the domain of f_0 , is an open and, possibly, bounded interval of \mathbb{R} containing 0. We also set

$$F_0(r) := \int_0^r f_0(s) \, ds, \quad F(x, r) := \int_0^r f(x, s) \, ds = F_0(r) - \lambda(x) \frac{r^2}{2}. \quad (2.5)$$

We assume J be a linear operator such that

$$J \in \mathcal{L}(L^p(\Omega), L^p(\Omega)), \quad \|J(u)\|_{L^p(\Omega)} \leq L\|u\|_{L^p(\Omega)}, \quad (2.6)$$

for some $L > 0$ independent of p and all $p \in [1, +\infty]$. Moreover we assume that

$$\exists p_* \in [1, +\infty) : \quad J \in \mathcal{L}(L^{p_*}(\Omega), L^\infty(\Omega)) \quad (2.7)$$

and, finally,

$$J \quad \text{is a compact self-adjoint operator from } H \text{ to } H. \quad (2.8)$$

Observe that the concrete form of the nonlocal operator J (see (1.5)) satisfies assumptions (2.6)-(2.8), provided that K is smooth enough.

We can then introduce the *energy functional*

$$\mathcal{E}(\vartheta, \chi) := \int_\Omega \left(\frac{1}{2} |\vartheta|^2 + F(\cdot, \chi) + \frac{1}{2} J[\chi]\chi \right). \quad (2.9)$$

It is immediate to realize that under the above assumptions \mathcal{E} could be unbounded from below. Thus, we need some condition implying that \mathcal{E} has some coercivity. In particular, we will consider two different situations. The first one deals with what we will call a *smooth* potential:

Assumption 2.1. We assume (2.4) with $I = \mathbb{R}$. Moreover, we ask that

$$\kappa_f |r|^{1+\epsilon} - c_f \leq f_0(r) \operatorname{sign} r \leq C_f (|r|^{1+\epsilon} + 1) \quad \forall r \in \mathbb{R} \quad (2.10)$$

and for some $\epsilon > 0$, $\kappa_f > 0$, $c_f \geq 0$, $C_f > 0$.

We will speak, instead, of *singular* potentials in the following case:

Assumption 2.2. We assume (2.4) with I an open and bounded interval of \mathbb{R} containing 0. Moreover, we ask that

$$\lim_{r \rightarrow \partial I} f_0(r) \operatorname{sign} r = +\infty. \quad (2.11)$$

Our first result deals with the case when F is smooth. In this situation, we define the *energy space* (i.e., the set of all (ϑ, χ) 's such that $\mathcal{E}(\vartheta, \chi)$ is finite), as the Banach space

$$\mathcal{X} := H \times L^{2+\epsilon}(\Omega). \quad (2.12)$$

Indeed, by Assumption 2.1, $\mathcal{E}(\vartheta, \chi)$ is finite if and only if $(\vartheta, \chi) \in \mathcal{X}$.

Theorem 2.3. *Let (2.6)-(2.8) and Assumption 2.1 hold and let $(\vartheta_0, \chi_0) \in \mathcal{X}$. Then, there exists one and only one couple (ϑ, χ) satisfying, for all $T > 0$,*

$$\vartheta \in H^1(0, T; V') \cap L^\infty(0, T; H) \cap L^2(0, T; V), \quad (2.13)$$

$$\chi \in H^1(0, T; H) \cap L^\infty(0, T; L^{2+\epsilon}(\Omega)), \quad (2.14)$$

solving, a.e. in $(0, T)$, system (2.1)-(2.2), and enjoying the initial conditions

$$\vartheta|_{t=0} = \vartheta_0, \quad \chi|_{t=0} = \chi_0, \quad \text{a.e. in } \Omega. \quad (2.15)$$

Moreover, there exist a time $T_0 \geq 0$ depending on the “initial energy” $\mathbb{E}_0 := \mathcal{E}(\vartheta_0, \chi_0)$ and a constant C_0 independent of \mathbb{E}_0 , such that

$$\|\vartheta(t)\|_{L^\infty(\Omega)} + \|\chi(t)\|_{L^\infty(\Omega)} + \|\chi_t(t)\|_{L^\infty(\Omega)} \leq C_0 \quad \forall t \geq T_0. \quad (2.16)$$

In the case when F is singular, the *energy space* is given by

$$\mathcal{X} := \{(\vartheta, \chi) \in H \times H : F_0(\chi) \in L^1(\Omega)\}. \quad (2.17)$$

Actually, since the domain I of F_0 is bounded, it is clear that $\mathcal{X} \subset H \times L^\infty(\Omega)$. Due to the constraint term F_0 , \mathcal{X} is not a linear space in this case. Nevertheless, it is easy to prove that has a complete metric structure with respect to a natural distance function (see, e.g., [28, Sec. 3] for details).

Theorem 2.4. *Let (2.6)-(2.8) and Assumption 2.2 hold and let $(\vartheta_0, \chi_0) \in \mathcal{X}$. Then, there exists one and only one couple (ϑ, χ) satisfying, for all $T > 0$,*

$$\vartheta \in H^1(0, T; V') \cap L^\infty(0, T; H) \cap L^2(0, T; V), \quad (2.18)$$

$$\chi \in H^1(0, T; H), \quad F_0(\chi) \in L^\infty(0, T; L^1(\Omega)), \quad (2.19)$$

$$f_0(\chi) \in L^2(0, T; H), \quad (2.20)$$

and solving (2.1)-(2.2) together with the initial conditions (2.15). Moreover, there exist a time $T_0 \geq 0$ depending on the “initial energy” $\mathbb{E}_0 := \mathcal{E}(\vartheta_0, \chi_0)$ and a constant C_0 independent of \mathbb{E}_0 , such that (2.16) holds, together with the separation property

$$\|f(\chi)\|_{L^\infty(\Omega)} \leq C_0 \quad \forall t \geq T_0. \quad (2.21)$$

A couple (ϑ, χ) in the condition either of Theorem 2.3 or of Theorem 2.4 will be called an *energy solution* in what follows.

Remark 2.5. An autonomous heat source term in equation (2.1) could be easily handled (see, e.g., [16]). Some care is required in the non-autonomous case (cf. [18] for local systems).

3 Proof of Theorem 2.3

This existence proof is a slight generalization of [15, Thm. 1.1]. Thus we will proceed formally with the a priori estimates. Uniqueness goes exactly as in [15], while we will give all the details about (2.16). In the sequel we will note with the letter c a generic positive constant, allowed to vary on occurrence, depending only on f_0 , λ and L (cf. (2.6)). In particular, we will always assume c to be independent of the initial data and of time. The letter κ will note the positive constants, depending on the same quantities as c , appearing in estimates from below.

Energy estimate. We test (2.1) by ϑ , (2.2) by χ_t and take the sum. This gives (recall that J is self-adjoint)

$$\frac{d}{dt}\mathcal{E}(\vartheta, \chi) + \|\nabla\vartheta\|^2 + \|\chi_t\|^2 \leq 0. \quad (3.1)$$

Next, we test (2.2) by χ , to obtain

$$\frac{1}{2}\frac{d}{dt}\|\chi\|^2 + (J[\chi], \chi) + (f(\cdot, \chi), \chi) \leq (\vartheta, \chi). \quad (3.2)$$

By (2.6), Assumption 2.1, and $\lambda \in L^\infty(\Omega)$, it is clear that

$$(f(\cdot, \chi), \chi) + (J[\chi], \chi) \geq \kappa \int_{\Omega} F(\cdot, \chi) + \kappa(J[\chi], \chi) - c \geq \kappa\|\chi\|_{2+\epsilon}^{2+\epsilon} + \kappa(J[\chi], \chi) - c. \quad (3.3)$$

Moreover, by Poincaré's and Young's inequalities,

$$(\vartheta, \chi) \leq \frac{1}{2}\|\nabla\vartheta\|^2 + c\|\chi\|^2 \leq \frac{1}{2}\|\nabla\vartheta\|^2 + \sigma\|\chi\|_{2+\epsilon}^{2+\epsilon} + c_\sigma, \quad (3.4)$$

for small $\sigma > 0$ and c_σ depending on σ . Then, summing (3.1) with (3.2) and taking (3.3), (3.4) into account, we arrive at

$$\frac{d}{dt}\left(\mathcal{E}(\vartheta, \chi) + \frac{1}{2}\|\chi\|^2\right) + \kappa(\mathcal{E}(\vartheta, \chi) + \|\nabla\vartheta\|^2 + \|\chi_t\|^2) \leq c. \quad (3.5)$$

Integrating (3.5), we then obtain

$$\mathcal{E}(\vartheta(t), \chi(t)) \leq c(\mathbb{E}_0 e^{-\kappa t} + 1), \quad (3.6)$$

for some new value of c , independent of T . In particular, by (2.10), this implies

$$\|\vartheta(t)\|^2 + \|\chi(t)\|_{2+\epsilon}^{2+\epsilon} \leq c(\mathbb{E}_0 e^{-\kappa t} + 1). \quad (3.7)$$

Moreover, integrating (3.1) over the generic interval (t, T) , and using (3.6), we infer

$$\int_t^T (\|\nabla\vartheta\|^2 + \|\chi_t\|^2) \leq c(\mathbb{E}_0 e^{-\kappa t} + 1). \quad (3.8)$$

Being c independent of T , the above bound can be rewritten also for $T = +\infty$.

Regularization estimates for ϑ . From (3.8) and the Poincaré inequality, it is clear that for any $s \in [0, +\infty)$ there exists $\tau = \tau(s) \in [s, s+1]$ such that

$$\|\vartheta(\tau)\|_V^2 \leq c(\mathbb{E}_0 e^{-\kappa\tau} + 1). \quad (3.9)$$

Then, taking $s \geq 0$ and correspondingly choosing $\tau \in [s, s+1]$ such that (3.9) holds, testing (2.1) by $-\Delta\vartheta$ and integrating over (τ, T) for a generic $T \geq \tau$, recalling (3.5), and using Hölder's and Young's inequalities, we obtain

$$\|\nabla\vartheta\|_{L^\infty(\tau, T; H)}^2 + \int_\tau^T \|\Delta\vartheta\|^2 \leq c(\mathbb{E}_0 e^{-\kappa\tau} + 1). \quad (3.10)$$

In particular, noting that the above holds at least for some $\tau \in (0, 1)$ (corresponding to the choice $s = 0$), we have

$$\|\vartheta\|_{L^\infty(t, T; V)}^2 + \int_t^T \|\Delta\vartheta\|^2 \leq c(\mathbb{E}_0 e^{-\kappa t} + 1), \quad \forall 1 \leq t \leq T, \quad (3.11)$$

still with c independent of T .

Regularization estimates for χ . We will now work on a generic time interval $(S, S+2)$ for $S \geq 1$. Then, as a consequence of estimate (3.10) and interpolation, we have that

$$\|\vartheta\|_{L^{10}((S, S+2) \times \Omega)} \leq C. \quad (3.12)$$

Here and in what follows, C will always denote a quantity of the form

$$C = Q(\mathbb{E}_0 e^{-\kappa S}), \quad (3.13)$$

where Q is a computable nonnegative-valued monotone function, whose expression is allowed to vary on occurrence, depending only on the fixed parameters of the system.

That said, we choose a sequence of small time steps τ_n , $n \in \mathbb{N}$, defined by

$$\tau_n = \frac{3}{2\pi^2 n^2} \quad \text{so that} \quad \sum_{n=1}^{\infty} \tau_n = \frac{1}{4} \quad (3.14)$$

and proceed by induction. Namely, we set $t_0 := S$ and assume that, for $n \geq 1$, given $t_{n-1} \geq S$, there exists $t_n \in (t_{n-1}, t_{n-1} + \tau_n)$ such that

$$\|\chi(t_n)\|_{L^{2+n\epsilon}(\Omega)}^{2+n\epsilon} \leq C\tau_n^{-1}. \quad (3.15)$$

Notice that this is surely true as $n = 1$ once one sets $t_0 = S$, thanks to (3.7). Then, we test (2.2) by $|\chi|^{n\epsilon}\chi$. In principle this would be not an admissible test function (actually at this level (2.2) makes sense just as a relation in $L^2((0, T) \times \Omega)$ and $|\chi|^{n\epsilon}\chi$ needs not have the L^2 -summability). However, the procedure could be easily justified by using a truncation of $|\chi|^{n\epsilon}\chi$ as a test function, and then passing to the limit w.r.t. the truncation parameter via the monotone convergence theorem (the details are left to the reader).

Moreover, since we only need a finite number of iterations, we will not take care of the dependence of the various constants on n and on ϵ . We infer

$$\frac{1}{2+n\epsilon} \frac{d}{dt} \|\chi\|_{2+n\epsilon}^{2+n\epsilon} + (f_0(\chi), |\chi|^{n\epsilon}\chi) \leq (\vartheta, |\chi|^{n\epsilon}\chi) + (-J[\chi] + \lambda(\cdot)\chi, |\chi|^{n\epsilon}\chi). \quad (3.16)$$

Then, by (2.10),

$$(f_0(\chi), |\chi|^{n\epsilon}\chi) \geq \kappa \|\chi\|_{2+(n+1)\epsilon}^{2+(n+1)\epsilon} - c. \quad (3.17)$$

Moreover,

$$\begin{aligned} (\vartheta, |\chi|^{n\epsilon}\chi) &\leq \| |\chi|^{n\epsilon}\chi \|_{\frac{2+(n+1)\epsilon}{1+n\epsilon}} \|\vartheta\|_{\frac{2+(n+1)\epsilon}{1+\epsilon}} \\ &\leq \|\chi\|_{2+(n+1)\epsilon}^{1+n\epsilon} \|\vartheta\|_{\frac{2+(n+1)\epsilon}{1+\epsilon}} \\ &\leq \frac{\kappa}{4} \|\chi\|_{2+(n+1)\epsilon}^{2+(n+1)\epsilon} + c \|\vartheta\|_{\frac{2+(n+1)\epsilon}{1+\epsilon}}^{\frac{2+(n+1)\epsilon}{1+\epsilon}}. \end{aligned} \quad (3.18)$$

Finally, by (2.6) and $\lambda \in L^\infty(\Omega)$,

$$(-J[\chi] + \lambda(\cdot)\chi, |\chi|^{n\epsilon}\chi) \leq c \| |\chi|^{n\epsilon}\chi \|_{\frac{2+(n+1)\epsilon}{1+n\epsilon}} \|\chi\|_{\frac{2+(n+1)\epsilon}{1+\epsilon}} \leq c \|\chi\|_{2+(n+1)\epsilon}^{2+n\epsilon} \leq \frac{\kappa}{4} \|\chi\|_{2+(n+1)\epsilon}^{2+(n+1)\epsilon} + c. \quad (3.19)$$

Then, integrating (3.16) over $(t_n, S+2)$, recalling (3.12), and using (3.15) and the induction hypothesis, we arrive at

$$\|\chi\|_{L^\infty(t_n, S+2; L^{2+n\epsilon}(\Omega))}^{2+n\epsilon} + \|\chi\|_{L^{2+(n+1)\epsilon}((t_n, S+2) \times \Omega)}^{2+(n+1)\epsilon} \leq C + c \|\chi(t_n)\|_{L^{2+n\epsilon}(\Omega)}^{2+n\epsilon} \leq C(1 + \tau_n^{-1}), \quad (3.20)$$

where the second term on the left hand side ensures that condition (3.15) will be fulfilled at the level $n+1$. In particular, the procedure can be iterated at least until n satisfies the constraint

$$\frac{2+(n+1)\epsilon}{1+\epsilon} \leq 10, \quad \text{i.e., } n \leq \frac{8+9\epsilon}{\epsilon}. \quad (3.21)$$

More precisely, since $(8 + 9\epsilon)/\epsilon$ may not be an integer, our method works at least until we reach some $n_{\max} \geq 8(1 + \epsilon^{-1})$. Thus, we obtain (at least) the bound

$$\|\chi\|_{L^\infty(S+1/4, S+2; L^{10+8\epsilon}(\Omega))} + \|\chi\|_{L^{10+9\epsilon}((S+1/4, S+2) \times \Omega)} \leq C, \quad (3.22)$$

where C additionally depends on the chosen sequence τ_n . The *upper* bound in hypothesis (2.10) then gives also

$$\|f(\cdot, \chi)\|_{L^9((S+1/4, S+2) \times \Omega)} \leq C, \quad (3.23)$$

whence, comparing terms in (2.2) and using (3.12), assumption (2.6) and (3.22)-(3.23), we also obtain

$$\|\chi_t\|_{L^9((S+1/4, S+2) \times \Omega)} \leq C. \quad (3.24)$$

With (3.24) at disposal (indeed, any exponent $p > 3$, in place of 9, would be sufficient for the purpose), we can apply to equation (2.1) (which is a linear PDE) a standard Alikakos-Moser iteration method [1] with time-smoothing (see, e.g., [29, Lemma 3.5], cf. also [26, Chap. III, Sec. 7]), to obtain

$$\|\vartheta\|_{L^\infty((S+1/2, S+2) \times \Omega)} \leq \Theta, \quad (3.25)$$

for some $\Theta > 0$ still having the form $\Theta = Q(\mathbb{E}_0 e^{-\kappa S})$.

Once we have (3.25), we can go back to (2.2) and perform further iterations of (3.16), restarting from $n = 0$ with the choice of $t_0 = S + 1/2$. Actually, (3.17) can still be used. On the other hand, thanks to (3.25), (3.18) can be now modified this way:

$$(\vartheta, |\chi|^{n\epsilon} \chi) \leq \| |\chi|^{n\epsilon} \chi \|_{\frac{2+(n+1)\epsilon}{1+n\epsilon}} \|\vartheta\|_\infty \leq \Theta \|\chi\|_{2+(n+1)\epsilon}^{1+n\epsilon} \leq \frac{\kappa}{4} \|\chi\|_{2+(n+1)\epsilon}^{2+(n+1)\epsilon} + c\Theta^{\frac{2+(n+1)\epsilon}{1+\epsilon}}. \quad (3.26)$$

At this point, we can proceed with the iterations exactly as before and notice that, still in a *finite* number of steps, we arrive at

$$\|\chi\|_{L^\infty(S+3/4, S+2; L^{p*}(\Omega))} \leq C. \quad (3.27)$$

Consequently, using property (2.7), we also obtain

$$\|J[\chi]\|_{L^\infty((S+3/4, S+2) \times \Omega)} \leq M, \quad (3.28)$$

for some constant $M > 0$ having the same dependence of C .

Remark 3.1. Unlike the case of parabolic type equations, it seems that, for equation (2.2), which has essentially an ODE structure, the Moser iteration method cannot be used to get directly an L^∞ -bound for χ . Actually, if we try to iterate the above procedure *infinitely many* times, we readily notice that the constants appearing on the right hand side's of the estimates (cf., e.g., (3.18) and (3.19)) would explode. This is due to the fact that, while in true parabolic equations the Moser exponents grow *exponentially* with respect to n , in the present case, the growth is just *linear* (and, hence, too slow to take the constants under control). This fact also forces us to assume that the kernel J has at least some regularizing effect (i.e., assumption (2.7)).

Once we have (3.25) and (3.28) at our disposal, we can apply a comparison principle to get the L^∞ -bound for χ . Namely, we have that, for $t \in (S + 3/4, S + 2)$ and a.e. $x \in \Omega$, there holds

$$\chi_t(t, x) + f_0(\chi(t, x)) = \lambda(x)\chi(t, x) + \vartheta(t, x) - J[\chi](t, x). \quad (3.29)$$

Let us now “freeze” the variable x . Actually, thanks to the first (2.19), the map $t \mapsto \chi(t, x)$ is (Lipschitz) continuous for a.e. $x \in \Omega$.

Let now $\Lambda > 0$ (to be chosen later) and set

$$\Lambda^+(x) := \{t \in (S + 3/4, S + 2) : \chi(t, x) \geq \Lambda\}. \quad (3.30)$$

Then, using (2.10), (3.25), and (3.28), (3.29) gives

$$\chi_t(t, x) + \kappa_f \chi(t, x)^{1+\epsilon} \leq \lambda(x)\chi(t, x) + \Theta + M + c_f. \quad (3.31)$$

for a.e. $x \in \Omega$ and all $t \in \Lambda^+(x)$. Hence, dividing by $\chi^{1+\epsilon}(t, x)$, we obtain

$$-\frac{1}{\epsilon} \frac{d}{dt} \chi^{-\epsilon}(t, x) + \kappa_f \leq \lambda(x) \chi^{-\epsilon}(t, x) + (\Theta + M + c_f) \chi^{-(1+\epsilon)}(t, x). \quad (3.32)$$

It is then clear that Λ can be taken large enough (in a way only depending on the L^∞ -norm of λ and on the known constants Θ , M , c_f and κ_f) so that, for $t \in \Lambda^+(x)$,

$$-\frac{1}{\epsilon} \frac{d}{dt} \chi^{-\epsilon}(t, x) + \frac{\kappa_f}{2} \leq 0. \quad (3.33)$$

In particular, for such times t , the function $t \mapsto \chi(t, x)$ is (strictly) decreasing.

This implies that, if $t \in (S + 3/4, S + 2)$ and $t \notin \Lambda^+(x)$, then $s \notin \Lambda^+(x)$ for all $s \in [t, S + 2)$. In other words, if $\chi(t, x)$ is smaller than Λ , it can never become larger than it. Thus, assuming that x is such that $\chi_S(x) := \chi(S + 3/4, x) \geq \Lambda$ (otherwise there is nothing to prove in view of the preceding discussion), integrating inequality (3.33) over $(S + 3/4, t)$, we obtain

$$\chi^{-\epsilon}(t, x) \geq \chi_S(x)^{-\epsilon} + \frac{\kappa_f \epsilon (t - S - 3/4)}{2}, \quad \text{where } \chi_S(x) := \chi(S + 3/4, x), \quad (3.34)$$

at least for all $t \geq S$ such that $\chi(t, x) \geq \Lambda$. Equivalently, we can write

$$\chi(t, x) \leq \left(\frac{2\chi_S^\epsilon(x)}{\kappa_f \epsilon \chi_S^\epsilon(x)(t - S - 3/4) + 2} \right)^{\frac{1}{\epsilon}}. \quad (3.35)$$

Consequently, it is clear that there exists $\Lambda' > 0$, *independent of the value of $\chi_S(x)$* , such that

$$\chi(t, x) \leq \max\{\Lambda, \Lambda'\} \quad \text{for almost all } (t, x) \in (S + 1, S + 2) \times \Omega. \quad (3.36)$$

For instance, one can take $\Lambda' = (8\kappa_f^{-1}\epsilon^{-1})^{1/\epsilon}$. Of course, a similar bound from below (of the form $\chi(t, x) \geq -\max\{\Lambda, \Lambda'\}$) can be proved in the same way. Thanks to the arbitrariness of the starting time $S \in [1, +\infty)$ and recalling once more (3.25), we have obtained the bounds for ϑ and χ in (2.16), for instance with the choice of $T_0 = 2$ (however, see Remark 4.2 below). Then, the remaining bound for χ_t follows by Assumption 2.1, (2.6), and a comparison of terms in (2.2). Theorem 2.3 is proved.

4 Proof of Theorem 2.4

This proof is presented in full detail (also for what concerns existence) since, to the best of our knowledge, this singular potential case has never been analyzed in the literature.

4.1 Approximation

We assume $I = \text{dom } f_0 = (-1, 1)$ for simplicity. We start by approximating the singular function f_0 by a sequence f_δ , $\delta \in (0, 1)$, such that

$$f_\delta \in C^1(\mathbb{R}; \mathbb{R}), \quad f'_\delta(r) \geq 0 \quad \forall r \in I, \quad f_\delta(0) = 0, \quad (4.1)$$

for all $\delta \in (0, 1)$. Moreover, we require that

$$\kappa_0 |r|^2 - c_0 \leq f_\delta(r) \text{ sign } r \leq C_\delta (|r|^2 + 1) \quad \forall r \in \mathbb{R}, \quad \delta \in (0, 1), \quad (4.2)$$

where the constants C_δ depend on δ (and, in fact, will explode as $\delta \searrow 0$), while κ_0 and c_0 are assumed to be independent of δ . In other words, we are asking that Assumption 2.1 holds with $\epsilon = 1$. The singular character of f_0 ensures that the lower bound in (4.2) can be assumed to hold uniformly in δ . Moreover, we assume that

$$f_{\delta_1}(r) \text{ sign } r \leq f_{\delta_2}(r) \text{ sign } r \quad \forall r \in I = (-1, 1), \quad \forall 0 < \delta_2 < \delta_1 < 1, \quad (4.3)$$

$$f_\delta(r) \rightarrow f_0(r) \quad \text{uniformly on compact sets of } I = (-1, 1), \quad (4.4)$$

whereas, for all $r \in \mathbb{R} \setminus (-1, 1)$, $f_\delta(r) \operatorname{sign} r \rightarrow +\infty$. We also set

$$F_\delta(r) := \int_0^r f_\delta(s) \, ds. \quad (4.5)$$

Actually, it is easy to check that, by (4.3), $F_{\delta_1} \leq F_{\delta_2}$ if $\delta_2 < \delta_1$. Moreover, we can assume that

$$f_\delta(r)r - \lambda(x)r^2 \geq \kappa F_\delta(r) - c \quad \forall r \in \mathbb{R}, \quad \delta \in (0, 1), \quad (4.6)$$

for suitable constants $c \geq 0$, $\kappa > 0$ independent of δ . The details of the construction of this approximating family are left to the reader. For instance, one possibility could be that of taking $f_\delta(r) := g_\delta(r) + \delta^{-1}((r - (1 - \delta))^+)^2$ for $r \geq 0$ (and an analogous choice for $r < 0$), where g_δ is the Yosida regularization of f_0 (cf., e.g., [7]).

We also notice that, if $\chi_0 \in \mathcal{X}$ (cf. (2.17)), recalling that $|\Omega| = 1$, we have

$$\|\chi_0\|_p \leq 1 \quad \forall p \in [1, \infty]. \quad (4.7)$$

Then, the above approximation permits to apply Theorem 2.3 to the system

$$\vartheta_{\delta,t} + \chi_{\delta,t} + A\vartheta_\delta = 0, \quad (4.8)$$

$$\chi_{\delta,t} + J[\chi_\delta] + f_\delta(\chi_\delta) = \vartheta_\delta + \lambda(x)\chi_\delta, \quad (4.9)$$

with the initial conditions (2.15) (note that the initial data are not approximated). This yields, for any $\delta \in (0, 1)$, a solution $(\vartheta_\delta, \chi_\delta)$ satisfying (2.13)-(2.14) (where $\epsilon = 1$, cf. (4.2)), together with (2.16). *A priori*, these regularity properties could depend on the approximation parameter δ . However, we shall see in a while that, in fact, δ -independent estimates are satisfied.

4.2 Uniform estimates and passage to the limit

In what follows we will assume that all constants κ, c are *independent* of δ . As we repeat the estimates performed in the proof of Theorem 2.3, it is easy to realize that, defining the approximate energy as

$$\mathcal{E}_\delta(\vartheta, \chi) := \int_\Omega \left(\frac{1}{2} |\vartheta|^2 + F_\delta(\chi) - \frac{\lambda(\cdot)}{2} \chi^2 + \frac{1}{2} J[\chi] \chi \right), \quad (4.10)$$

the function \mathcal{E}_δ satisfies (3.5) with κ, c independent of δ . Then, noting that

$$\mathbb{E}_{0,\delta} := \mathcal{E}_\delta(\vartheta_0, \chi_0) = \int_\Omega \left(\frac{1}{2} |\vartheta_0|^2 + F_\delta(\chi_0) - \frac{\lambda(\cdot)}{2} \chi_0^2 + \frac{1}{2} J[\chi_0] \chi_0 \right) \leq \mathbb{E}_0 \quad (4.11)$$

thanks to $F_\delta \leq F_0$, it is easy to check that the “Energy estimate” of the previous section can be repeated to obtain relations analogue to (3.6), (3.7), (3.8), that hold now uniformly w.r.t. δ . Moreover, we can test (4.9) by $f_\delta(\chi_\delta)$ and use (3.7), (3.8), and the properties of J to infer

$$\|f_\delta(\chi_\delta)\|_{L^2(t, t+1; H)} \leq Q(\mathbb{E}_0 e^{-\kappa t}), \quad \forall t \geq 0, \quad (4.12)$$

with Q independent of δ . We now show that the estimates detailed above suffice to take the limit $\delta \searrow 0$. Actually, (3.6)-(3.8) and (4.12) guarantee that, for any $T > 0$,

$$\vartheta_\delta \rightarrow \vartheta \quad \text{weakly star in } H^1(0, T; V') \cap L^\infty(0, T; H) \cap L^2(0, T; V), \quad (4.13)$$

$$\chi_\delta \rightarrow \chi \quad \text{weakly in } H^1(0, T; H), \quad (4.14)$$

$$f_\delta(\chi_\delta) \rightarrow \overline{f_0(\chi)} \quad \text{weakly in } L^2(0, T; H). \quad (4.15)$$

Here and below, we adopt the convention of overlining unidentified weak limits. Thanks to linearity and continuity of operator \mathcal{J} , it is then easy to show that, at the limit $\delta \searrow 0$,

$$\vartheta_t + \chi_t + A\vartheta = 0, \quad (4.16)$$

$$\chi_t + J[\chi] + \overline{f_0(\chi)} - \lambda(x)\chi = \vartheta \quad (4.17)$$

and the initial conditions (2.15) are satisfied as well.

Then, to conclude the proof, we have to show the identification $\overline{f_0(\chi)} = f_0(\chi)$ almost everywhere in $(0, T) \times \Omega$. To do this, we follow with some variations the argument given in [15, Sec. 2.3], which we report for the reader's convenience.

First of all, letting

$$\omega_\delta : [0, T] \rightarrow \mathbb{R}, \quad \omega_\delta(t) := \|\chi_\delta(t) - \chi(t)\|^2, \quad (4.18)$$

using (4.14) it is a standard check to verify that

$$\|\omega_\delta\|_{H^1(0, T)} \leq c. \quad (4.19)$$

Hence, we can assume that (here and below, all convergence relations are intended up to extraction of non-relabelled subsequences)

$$\omega_\delta \rightarrow \omega \quad \text{strongly in } C^0([0, T]), \quad (4.20)$$

where ω is continuous and nonnegative. Now, as a further consequence of (4.14), we have that

$$\chi_\delta \rightarrow \chi \quad \text{in } C_w([0, T]; H). \quad (4.21)$$

In particular, for all $t \in [0, T]$, $\chi_\delta(t)$ converges to $\chi(t)$ weakly in H . Next, we compute the difference between (4.9) and (4.17), test it $\chi_\delta - \chi$, and integrate with respect to the space variables. This gives

$$\frac{1}{2} \frac{d}{dt} \|\chi_\delta - \chi\|^2 + (f_\delta(\chi_\delta) - \overline{f_0(\chi)}, \chi_\delta - \chi) = (\vartheta_\delta - J[\chi_\delta] + \lambda(\cdot)\chi_\delta - \vartheta + J[\chi] - \lambda(\cdot)\chi, \chi_\delta - \chi). \quad (4.22)$$

Now, we notice that, by (4.12), (4.14) and the first inequality in (4.2),

$$\|f_\delta(\chi_\delta)\chi_\delta\|_{L^{4/3}((0, T) \times \Omega)} \leq c. \quad (4.23)$$

Consequently,

$$f_\delta(\chi_\delta)\chi_\delta \rightarrow \overline{f_0(\chi)\chi} \quad \text{weakly in } L^{4/3}((0, T) \times \Omega). \quad (4.24)$$

By definition of subdifferential, we have, almost everywhere in $(0, T) \times \Omega$,

$$f_\delta(\chi_\delta)(\chi_\delta - \chi) \geq F_\delta(\chi_\delta) - F_\delta(\chi) \geq F_\delta(\chi_\delta) - F_0(\chi). \quad (4.25)$$

Let us now test (4.25) by a nonnegative test function $\phi \in \mathcal{D}((0, T) \times \Omega)$ and integrate. Then, by convexity and lower semicontinuity of the functional

$$L^2((0, T) \times \Omega) \rightarrow \mathbb{R}, \quad v \mapsto \iint_{(0, T) \times \Omega} F_0(v)\phi, \quad (4.26)$$

using (4.15) and (4.24), we obtain that

$$\begin{aligned} & \iint_{(0, T) \times \Omega} \overline{f_0(\chi)\chi}\phi - \iint_{(0, T) \times \Omega} \overline{f_0(\chi)}\chi\phi \\ &= \lim_{\delta \searrow 0} \iint_{(0, T) \times \Omega} f_\delta(\chi_\delta)(\chi_\delta - \chi)\phi \\ &\geq \liminf_{\delta \searrow 0} \iint_{(0, T) \times \Omega} (F_\delta(\chi_\delta) - F_0(\chi))\phi \geq 0. \end{aligned} \quad (4.27)$$

To deduce the last inequality we have used the fact that the family of functionals $\{F_\delta\}$, being monotone increasing with respect to δ going to 0, converges to F_0 in the sense of Mosco (see, e.g., [3]) in $L^2((0, T) \times \Omega)$. In particular, we used here the \liminf -property of Mosco-convergence:

$$\iint_{(0, T) \times \Omega} F_0(v)\phi \leq \liminf_{\delta \searrow 0} \iint_{(0, T) \times \Omega} F_\delta(v_\delta)\phi \quad \text{for all } v_\delta \rightarrow v \text{ weakly in } L^2((0, T) \times \Omega). \quad (4.28)$$

Thus, we have, almost everywhere in $(0, T) \times \Omega$,

$$\overline{f_0(\chi)\chi} \geq \overline{f_0(\chi)}\chi. \quad (4.29)$$

Moreover, we notice that, thanks to (4.13), (4.14), the Aubin-Lions Lemma, and assumption (2.8),

$$(\overline{f_0(\chi)} + \vartheta_\delta - J[\chi_\delta] - \vartheta + J[\chi], \chi_\delta - \chi) \rightarrow 0, \quad (4.30)$$

at least in the sense of distributions over $(0, T)$.

Then, we can take the limit, as $\delta \searrow 0$, of (4.22). Using (4.24) and (4.29), and noting that the time-derivative operator is linear and continuous with respect to distributional convergence, we then obtain

$$\frac{1}{2} \frac{d}{dt} \lim_{\delta \searrow 0} \|\chi_\delta - \chi\|^2 \leq c_\lambda \lim_{\delta \searrow 0} \|\chi_\delta - \chi\|^2, \quad (4.31)$$

or, equivalently,

$$\frac{1}{2} \frac{d}{dt} \omega(t) \leq c_\lambda \omega(t). \quad (4.32)$$

Since ω is nonnegative and $\omega(0) = 0$, we then obtain $\omega(t) = 0$ for all $t \in [0, T]$. In other words,

$$\chi_\delta(t) \rightarrow \chi(t) \quad \text{strongly in } H, \quad \forall t \in [0, T]. \quad (4.33)$$

This fact, combined with (4.14) gives

$$\chi_\delta \rightarrow \chi \quad (\text{at least}) \text{ strongly in } L^2(0, T; H), \quad (4.34)$$

which entails in particular $\overline{f_0(\chi)} = f_0(\chi)$. Thus, (4.17) reduces to (2.2), as desired.

4.3 Regularization estimates and separation property

The above procedure is sufficient to get existence of an energy solution to our system under Assumption 2.2. Uniqueness of this solution is proved as in the other cases.

To prove rigorously the separation property (2.21), we go back to the δ -system (4.8)-(4.9) and start by noticing that the regularization estimates of Section 3 hold *uniformly* in δ . Actually, concerning the ‘‘Regularization estimates for ϑ ’’ it is easy to see that nothing changes and the analogue of (3.9)-(3.11) hold uniformly in δ . Concerning the ‘‘Regularization estimates for χ ’’, we can proceed as before (where we now have $\epsilon = 1$, of course), until we reach estimate (3.22). Indeed, in this part of the Moser iteration, we only use the estimates (3.7)-(3.8) and (3.9)-(3.11), which are uniform in δ , and the estimate from below (i.e., the first inequality) in (4.2), which is also independent of δ . Consequently, we now have the analogue of (3.22), which, for $\epsilon = 1$ and in the current notation, becomes

$$\|\chi_\delta\|_{L^\infty(S+1/4, S+2; L^{18}(\Omega))} + \|\chi_\delta\|_{L^{19}((S+1/4, S+2) \times \Omega)} \leq C. \quad (4.35)$$

Here and in what follows, all constants c , κ and C have the same meaning as in the previous section and, in addition, are assumed to be independent of δ .

However, at this point we can no longer deduce the analogue of (3.23) directly, since this requires use of the *upper* bound in (4.2), where the constants *do depend* on δ .

We then have to proceed with some more care and set, for $p \in [1, \infty)$,

$$\phi_\delta^p(s) := \int_0^s |f_\delta(r)|^p \operatorname{sign} r \, dr \leq |f_\delta(s)|^p |s|. \quad (4.36)$$

Then, we test (4.9) by $|f_\delta(\chi_\delta)|^p \operatorname{sign} \chi_\delta$, with p to be chosen below. This gives

$$\begin{aligned} \frac{d}{dt} \int_\Omega \phi_\delta^p(\chi_\delta) + \|f_\delta(\chi_\delta)\|_{p+1}^{p+1} &= (|f_\delta(\chi_\delta)|^p \operatorname{sign} \chi_\delta, \vartheta_\delta + \lambda(\cdot)\chi_\delta - J[\chi_\delta]) \\ &\leq \frac{1}{2} \|f_\delta(\chi_\delta)\|_{p+1}^{p+1} + c(\|\vartheta_\delta\|_{p+1}^{p+1} + \|\chi_\delta\|_{p+1}^{p+1}). \end{aligned} \quad (4.37)$$

Thus, integrating in time over (τ, t) , where $S + 1/4 \leq \tau \leq t \leq S + 2$, we arrive at

$$\begin{aligned} & \int_{\Omega} \phi_{\delta}^p(\chi_{\delta}(t)) + \frac{1}{2} \|f_{\delta}(\chi_{\delta})\|_{L^{p+1}((\tau, t) \times \Omega)}^{p+1} \\ & \leq \int_{\Omega} \phi_{\delta}^p(\chi_{\delta}(\tau)) + c(\|\vartheta_{\delta}\|_{L^{p+1}((\tau, t) \times \Omega)}^{p+1} + \|\chi_{\delta}\|_{L^{p+1}((\tau, t) \times \Omega)}^{p+1}). \end{aligned} \quad (4.38)$$

Let us now first choose $p = 5/3$. Then, according to (4.12), we can take $\tau \in (S + 1/4, S + 5/16)$ such that

$$\|f_{\delta}(\chi_{\delta}(\tau))\|_2^2 \leq C. \quad (4.39)$$

Then, by the inequality in (4.36),

$$\int_{\Omega} \phi_{\delta}^{5/3}(\chi_{\delta}(\tau)) \leq \| |f_{\delta}(\chi_{\delta}(\tau))|^{5/3} \|_{6/5} \|\chi_{\delta}(\tau)\|_6 \leq \|f_{\delta}(\chi_{\delta}(\tau))\|_2^{5/3} \|\chi_{\delta}(\tau)\|_6 \quad (4.40)$$

and both factors on the right hand side are controlled, uniformly in δ , thanks to (4.39) and (4.35), respectively. Then, noting that the other terms on the right hand side of (4.38) are estimated due to (3.12) and (4.35), we readily infer

$$\|f_{\delta}(\chi_{\delta})\|_{L^{8/3}((S+5/16, S+2) \times \Omega)} \leq C. \quad (4.41)$$

At this point, we iterate the procedure by choosing now $p = 7/3$ in (4.38), taking $\tau \in (S+5/16, S+3/8)$ such that the analogue of (4.39) with $8/3$ in place of 2 holds (this is possible thanks to (4.41)), and replacing (4.40) with

$$\int_{\Omega} \phi_{\delta}^{7/3}(\chi_{\delta}(\tau)) \leq \| |f_{\delta}(\chi_{\delta}(\tau))|^{7/3} \|_{8/7} \|\chi_{\delta}(\tau)\|_8 \leq \|f_{\delta}(\chi_{\delta}(\tau))\|_{8/3}^{7/3} \|\chi_{\delta}(\tau)\|_8. \quad (4.42)$$

Thus, we finally arrive at

$$\|f_{\delta}(\chi_{\delta})\|_{L^{10/3}((S+3/8, S+2) \times \Omega)} \leq C. \quad (4.43)$$

A comparison of terms in (4.9) permits now to get (in place of (3.24))

$$\|\chi_{\delta, t}\|_{L^{10/3}((S+3/8, S+2) \times \Omega)} \leq C \quad (4.44)$$

and the exponent $10/3$ is still sufficient to operate the Moser iteration argument with smoothing for ϑ . Thus, we arrive also in this case at the analogue of relation (3.25), with Θ independent of δ .

With this relation at disposal, we can repeat the ODE argument of Section 3, with essentially no variation. Actually, it is sufficient to use the lower bound in (4.2), which is uniform in δ . Summarizing, we have obtained

$$\|\vartheta_{\delta}\|_{L^{\infty}((1, \infty) \times \Omega)} \leq \Theta, \quad \|\chi_{\delta}\|_{L^{\infty}((1, \infty) \times \Omega)} \leq M, \quad (4.45)$$

with constants Θ and M independent of δ .

Remark 4.1. It is worth noting that, at the limit step, we have for free that $-1 \leq \chi \leq 1$ almost everywhere (and starting from $t = 0$), since F_0 is singular. However, (4.45) says something more, i.e., that we have, for $t \geq 1$, a uniform L^{∞} -bound independent of the approximation parameter for *both* components of the approximate solution. This is a nontrivial information especially as far as ϑ_{δ} is concerned (indeed, we just know that the initial datum ϑ_0 lies in H).

As a consequence of (4.45), there exists (a new) $\Lambda > 0$ depending only on λ , J , Θ and M , such that

$$\chi_{\delta, t}(t, x) + f_{\delta}(\chi_{\delta}(t, x)) \leq \Lambda \quad \text{for a.e. } (t, x) \in (1, \infty) \times \Omega. \quad (4.46)$$

Moreover,

$$\chi_{\delta}(1, x) \leq M \quad \text{a.e. in } \Omega. \quad (4.47)$$

Let now $\varepsilon \in (0, 1)$ such that $f_0(1 - \varepsilon) = 3\Lambda$ (cf. (2.11)). Then, there exists $\delta_0 \in (0, 1)$ such that $f_{\delta}(1 - \varepsilon) \geq 2\Lambda$ for all $\delta \in [0, \delta_0]$. In particular, by monotonicity (cf. (4.3)) we have that $f_{\delta}(r) \geq 2\Lambda$

for all $r \geq 1 - \varepsilon$ and $\delta \in (0, \delta_0]$. We now claim that there exists a time T_0 independent of δ such that, for (almost) all $x \in \Omega$, all $t \geq T_0$, and all $\delta \in (0, \delta_0]$, there holds that $\chi_\delta(t, x) \leq 1 - \varepsilon$. Indeed, thanks to the above discussion, we have that, if for some $\delta \in (0, \delta_0]$, $t \geq 1$, and $x \in \Omega$, it is $\chi_\delta(t, x) > 1 - \varepsilon$, then $t \mapsto \chi_\delta(t, x)$ is decreasing. Thus, once $t \mapsto \chi_\delta(t, x)$ enters the region where $\chi_\delta(t, x) \leq 1 - \varepsilon$, it never exits from it. Thus, freezing x as before and assuming that $\chi_\delta(1, x) > 1 - \varepsilon$ (otherwise there is nothing to prove), from (4.46) we have that

$$\chi_{\delta,t}(t, x) \leq -\Lambda, \quad (4.48)$$

at least as long as $\chi(t, x)$ remains larger than $1 - \varepsilon$. Then, integrating (4.46) in time and using (4.47), we infer

$$\chi_\delta(t, x) \leq \chi_\delta(1, x) - \Lambda(t - 1) \leq M - \Lambda(t - 1), \quad (4.49)$$

whence

$$\chi_\delta(t, x) \leq 1 - \varepsilon \quad \text{for all } t \geq \frac{M - (1 - \varepsilon)}{\Lambda} + 1 \quad (4.50)$$

and this bound is uniform with respect to δ . Proving the lower bound in the same way and passing to the limit $\delta \searrow 0$, we finally obtain (2.21), as desired.

Remark 4.2. Looking at the statement (and at the proof) of (2.21) one could think that the separation property occurs only after some waiting time. However, the property was given in that form just for the sake of simplicity. Indeed, refining a bit the arguments in the proof, it is easy to demonstrate that (2.21) is in fact an instantaneous property. Namely, there holds

$$-1 + \varepsilon(\tau) \leq \chi(t, x) \leq 1 - \varepsilon(\tau) \quad \text{for all } (t, x) \in (\tau, +\infty) \times \Omega \text{ and all } \tau > 0, \quad (4.51)$$

where $\varepsilon(\tau)$ goes to 0 as $\tau \searrow 0$. The details are left to the reader.

5 Global attractors

Here we establish the existence of a finite-dimensional global attractor for both smooth and singular potentials. The technique is the same as the one used in [16, Proof of Thm. 4.1] for singular unbounded potentials. However, here we start from very general initial data in the energy space and we exploit the previous regularization results to define a semigroup acting on a convenient invariant set.

Let us consider the case of smooth potentials first. From Theorem 2.3 it is clear that we can define a semigroup $S(t) : \mathcal{X} \rightarrow \mathcal{X}$ (cf. (2.12)) by setting $(\vartheta(t), \chi(t)) := S(t)(\vartheta_0, \chi_0)$, where (ϑ, χ) is the unique (energy) solution to (2.1)-(2.2), (2.15). Note that $\vartheta \in C^0([0, +\infty); H)$ while $\chi \in C^0([0, +\infty); H) \cap C_w^0([0, +\infty); L^{2+\epsilon}(\Omega))$. Moreover, on account of the Lipschitz continuity estimate [15, (1.11)], this semigroup is also closed in the sense of [27]. Thanks to (2.16) and (3.11), $S(t)$ has an absorbing set \mathcal{B} which is bounded in $(V \cap L^\infty(\Omega)) \times L^\infty(\Omega)$. Hence we can find $t_0 \geq 0$ such that $S(t)\mathcal{B} \subset \mathcal{B}$ for all $t \geq t_0$. Then, without loss of generality, we can suppose that $t_0 = 0$ and assume that \mathcal{B} is an invariant set for $S(t)$. Moreover, we can endow \mathcal{B} with the $V \times H$ -metric and obtain a complete metric space X .

We can then prove the following

Theorem 5.1. *Let the assumptions of Theorem 2.3 hold and suppose that*

$$\lambda_0 := \operatorname{ess\,sup}_{x \in \Omega} \lambda(x) < 0. \quad (5.1)$$

Then the dynamical system $(X, S(t))$ has a finite-dimensional connected global attractor.

For readers' convenience, we report here below the argument of [16].

PROOF. Consider $(\vartheta_{0i}, \chi_{0i}) \in X$, $i = 1, 2$, set

$$(\vartheta(t), \chi(t)) = ((\vartheta_1 - \vartheta_2)(t), (\chi_1 - \chi_2)(t))$$

where $(\vartheta_i(t), \chi_i(t)) = S(t)(\vartheta_{0i}, \chi_{0i})$ for $t \geq 0$, and observe that

$$\vartheta_t + \chi_t + A\vartheta = 0, \quad \text{a.e. in } (0, +\infty) \times \Omega, \quad (5.2)$$

$$\chi_t + J[\chi] + f(\cdot, \chi_1) - f(\cdot, \chi_2) = \vartheta, \quad \text{a.e. in } (0, +\infty) \times \Omega. \quad (5.3)$$

Let us multiply equation (5.2) by $A\vartheta(t)$. Integrating over Ω , we get

$$\frac{1}{2} \frac{d}{dt} \|\nabla \vartheta\|^2 + \|A\vartheta\|^2 = -(\chi_t, A\vartheta),$$

from which, using the Young and Poincaré inequalities, we derive the estimate

$$\frac{d}{dt} \|\nabla \vartheta\|^2 + \kappa_1 \|\nabla \vartheta\|^2 \leq \|\chi_t\|^2.$$

and, by comparison in (5.3), we deduce

$$\frac{d}{dt} \|\nabla \vartheta\|^2 + \kappa_1 \|\nabla \vartheta\|^2 \leq c_1 (\|\chi\|^2 + \|J[\chi]\|^2 + \|\vartheta\|^2), \quad (5.4)$$

for all $t \geq 0$. On the other hand, multiplying (5.3) by $\chi(t)$ and integrating over Ω we find

$$\frac{1}{2} \frac{d}{dt} \|\chi\|^2 + (f(\cdot, \chi_1) - f(\cdot, \chi_2), \chi) = -(J[\chi], \chi) + (\vartheta, \chi).$$

Thus, on account of (2.3), (2.4) and (5.1), we get

$$\frac{1}{2} \frac{d}{dt} \|\chi\|^2 + \lambda_0 \|\chi\|^2 \leq -(J[\chi], \chi) + (\vartheta, \chi).$$

Then, Young's inequality gives

$$\frac{1}{2} \frac{d}{dt} \|\chi\|^2 + \frac{\lambda_0}{2} \|\chi\|^2 \leq c_{\lambda_0} (\|J[\chi]\|^2 + \|\vartheta\|^2).$$

Since J is compact and self-adjoint on H , we can find a finite-dimensional projector Π_{λ_0} such that

$$\|J[v]\|^2 \leq \frac{\lambda_0}{4c_{\lambda_0}} \|v\|^2 + c \|\Pi_{\lambda_0}[v]\|^2, \quad (5.5)$$

for all $v \in H$. As a consequence we have

$$\frac{1}{2} \frac{d}{dt} \|\chi\|^2 + \frac{\lambda_0}{4} \|\chi\|^2 \leq c_{\lambda_0} (\|\Pi[\chi]\|^2 + \|\vartheta\|^2). \quad (5.6)$$

Adding inequality (5.4) multiplied by $\mu = \frac{\lambda_0}{16c_1}$ to (5.6) and using the analogue of (5.5) to estimate the term $\|J[\chi]\|^2$ on the right hand side of (5.4), we infer

$$\frac{d}{dt} \left(\mu \|\nabla \vartheta\|^2 + \frac{1}{2} \|\chi\|^2 \right) + \kappa_1 \mu \|\nabla \vartheta\|^2 + \frac{\lambda_0}{8} \|\chi\|^2 \leq c_{\lambda_0} (\|\Pi[\chi]\|^2 + \|\vartheta\|^2). \quad (5.7)$$

Therefore, from (5.7), we deduce

$$\begin{aligned} \|\vartheta(t)\|_V^2 + \|\chi(t)\|^2 &\leq c_{\lambda_0} e^{-\kappa_{\lambda_0} t} (\|\vartheta(0)\|_V^2 + \|\chi(0)\|^2) \\ &\quad + c_{\lambda_0} \int_0^t (\|\vartheta(\tau)\|^2 + \|\Pi_{\lambda_0}[\chi(\tau)]\|^2) d\tau, \end{aligned} \quad (5.8)$$

for all $t \in [0, T]$ and any fixed $T > 0$.

We now introduce the following pseudometric in X

$$\mathbf{d}_T((\vartheta_{01}, \chi_{01}), (\vartheta_{02}, \chi_{02})) = \left(\int_0^T (\|\vartheta_1 - \vartheta_2(\tau)\|^2 + \|\Pi_{\lambda_0}[(\chi_1 - \chi_2)(\tau)]\|^2) d\tau \right)^{1/2}$$

and we recall that a pseudometric is (pre)compact in X (with respect to the topology induced by the X -metric) if any bounded sequence in X contains a Cauchy subsequence with respect to \mathbf{d}_T (see, for instance, [21, Def. 2.57]).

Let us prove that \mathbf{d}_T is precompact in X . Let $\{(\vartheta_{0n}, \chi_{0n})\} \subset X$ (X is bounded) and set $(\vartheta_n(t), \chi_n(t)) = S(t)(\vartheta_{0n}, \chi_{0n})$. Thanks to (2.13), we have that $\{\vartheta_n\}$ is bounded in $L^2(0, T; D(A)) \cap H^1(0, T; H)$. Therefore it contains a subsequence which strongly converges in $L^2(0, T; V)$. On the other hand, we have that $\{\Pi_{\lambda_0}[\chi_n]\}$ is bounded in $L^\infty(0, T; H)$. Also, by comparison in (2.2), we deduce that $\{(\chi_n)_t\}$ is bounded in $L^\infty(0, T; H)$. Therefore $\{(\Pi_{\lambda_0}[\chi_n])_t\}$ is bounded in $L^\infty(0, T; H)$ as well. Then $\{\Pi_{\lambda_0}[\chi_n(\cdot)]\}$ contains a subsequence which strongly converges in $L^2(0, T; H)$. Summing up $\{(\vartheta_{0n}, \chi_{0n})\}$ contains a Cauchy subsequence with respect to \mathbf{d}_T .

From (5.8), we deduce that there exists $t^* > 0$ such that

$$\begin{aligned} & \|S(t^*)(\vartheta_{01}, \chi_{01}) - S(t^*)(\vartheta_{02}, \chi_{02})\|_X \\ & \leq \frac{1}{2} \|(\vartheta_{01} - \vartheta_{02}, \chi_{01} - \chi_{02})\|_X + \bar{c}_{\lambda_0} \mathbf{d}_{t^*}((\vartheta_{01}, \chi_{01}), (\vartheta_{02}, \chi_{02})). \end{aligned}$$

Hence $S(t)$ has a (connected) global attractor (see [21, Thm. 2.56, Prop. 2.59]) of finite fractal (i.e., box counting) dimension (cfr. [20, Thm. 2.8.1]). \blacksquare

In the case of singular potentials, we can first notice that, by (2.20) and the first (2.19), $t \mapsto \int_{\Omega} F_0(\chi(t))$ is absolutely continuous in $[0, T]$ due to [7, Lemma 3.3, p. 73]. Hence, taking \mathcal{X} defined by (2.17) as phase space, the semigroup $S(t)$ defined as above takes \mathcal{X} to itself for all $t \geq 0$. Using again (2.16) and on account of the separation property (2.21), it is not difficult to realize that there exists an absorbing set of the following form (note that (3.11) still holds):

$$\mathcal{B}(R, \beta) := \{(\vartheta, \psi) : \|\vartheta\|_V \leq R, -1 + \beta \leq \psi \leq 1 - \beta \text{ a.e. in } \Omega\}, \quad (5.9)$$

for a suitable pair of constants $(R, \beta) \in (0, +\infty) \times (0, 1)$. Then, reasoning as above, we can suppose that $\mathcal{B}(R, \beta)$ is invariant for $S(t)$ and we can endow it with the $V \times H$ -metric. The resulting complete metric space X is now our phase space and we have

Theorem 5.2. *Let the assumptions of Theorem 2.4 and (5.1) hold. Then, the dynamical system $(X, S(t))$ has a finite dimensional connected global attractor.*

The proof goes as above.

Remark 5.3. Assumption (5.1) is crucial for investigating the asymptotic behavior (see, e.g., [15, Thm. 1.2, (1.19) and Rem. 3], [6, Thm. 4.6 and (A4)] and [13, Ass. 4, Sec. 2 and Lemma 3.1]). Also, we recall that if f_0 is real analytic then, following [15, 19], one can prove that the ω -limit set of any pair (ϑ_0, χ_0) in the energy space \mathcal{X} reduces to a singleton $\{(0, \chi_\infty)\}$, where χ_0 solves the stationary problem

$$J[\chi_\infty] + f_0(\chi_\infty) - \lambda(x)\chi_\infty = 0, \quad \text{a.e. in } \Omega.$$

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